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A COMMENT ON: STRESS SINGULARITIES IN LAMINATED COMPOSITE WEDGES

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Recently, Ojikutu *et al.*[1] described a numerical method for determining the order λ of Williams-type stress singularities arising within the framework of classical lamination theory at the apex of a composite laminated wedge. After rewriting constitutive and equilibrium equations in polar coordinates, substitution of a displacement field exhibiting an r^λ behaviour yields a set of ordinary differential equations with variable coefficients depending upon the polar coordinate θ. The authors assume that a closed form solution does not exist and therefore they use a finite difference scheme based on a division of the apex angle into $n+1$ parts. After appropriate treatment of boundary conditions (attention was restricted to the case of a simply supported wedge, the support being free to move in the plane of the plate) this scheme leads to a homogeneous system of $n+2$ algebraic equations for the case of bending in symmetrical laminates and $3n+6$ equations for the case of coupled extension and bending in antisymmetrical laminates. The order λ of the singularity is then found as a root of the determinant of the algebraic system, taking $n = 24$ in the former case and $n = 18$ in the latter.

It is the purpose of this note to show that the cumbersome derivation of ordinary differential equations, boundary conditions and finite difference approximations may be avoided altogether because the partial differential equations of classical lamination theory, written in Cartesian coordinates, admit a general integral in closed form, similar to Lekhnitskii's approach[3] with complex-variable stress functions. This approach has been extensively used by Wang and co-workers[4–6] in stress singularity problems.

Constitutive and equilibrium equations of classical lamination theory are given by Jones[2] as

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{Bmatrix} u \\ w \end{Bmatrix} \tag{1}$$

$$\begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}^{\mathrm{T}} \begin{Bmatrix} N \\ M \end{Bmatrix} = \{\{0\} \tag{2}$$

where $\{u\}$ is the in-plane displacement vector, w is the deflection, $\{N\}$ and $\{M\}$ are stress resultants; $[A]$, $[D]$ and $[B]$ are respectively the extensional, bending and coupling stiffness matrices and $[L_1]$ and $\{L_2\}$ are partial differential operators

$$\{u\} = \{u_x \quad u_y\}^{\mathrm{T}} \tag{3}$$

$$\{N\} = \{N_{xx} \quad N_{yy} \quad N_{xy}\}^{\mathrm{T}} \tag{4}$$

$$\{M\} = \{M_{xx} \quad M_{yy} \quad M_{xy}\}^{\mathrm{T}} \tag{5}$$

$$[L_1] = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}^{\mathrm{T}} \tag{6}$$

$$\{L_2\} = \left\{\frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad 2\frac{\partial^2}{\partial x \partial y}\right\}^{\mathrm{T}} \tag{7}$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \tag{8}$$

with similar index numbering for [B] and [D]. Substitution of eqn (1) in eqn (2) yields

$$[P]\left\{\frac{u}{w}\right\} = \{0\} \tag{9}$$

where

$$[P] = \left[\begin{array}{c|c} L_1^{\mathrm{T}} A L_1 & L_1^{\mathrm{T}} B L_2 \\ \hline L_2^{\mathrm{T}} B L_1 & L_2^{\mathrm{T}} D L_2 \end{array}\right]. \tag{10}$$

The general integral of the partial differential equations (9) may be written in terms of arbitrary functions $\phi_k(z_k)$ with complex argument $z_k = x + \mu_k y$ as

$$\{u\} = \mathrm{Re} \sum_{k=1}^{4} \{U_k\} \frac{\mathrm{d}\phi_k}{\mathrm{d}z_k} \tag{11}$$

$$w = \mathrm{Re} \sum_{k=1}^{4} W_k \phi_k \tag{12}$$

where (U_k, W_k) is a non-trivial solution of the homogeneous algebraic system

$$[p(\mu_k)]\left\{\frac{U_k}{W_k}\right\} = \{0\} \tag{13}$$

and μ_k $(k = 1, \ldots, 4)$ are the complex roots with positive imaginary part of the determinant of $[p(\mu)]$. This matrix is obtained from eqns (6), (7) and (10) by formally replacing $(\partial/\partial x, \partial/\partial y)$ by $(1, \mu)$

$$[p(\mu)] = \left[\begin{array}{c|c} l_1^{\mathrm{T}} A\, l_1 & l_1^{\mathrm{T}} B\, l_2 \\ \hline l_2^{\mathrm{T}} B\, l_1 & l_2^{\mathrm{T}} D\, l_2 \end{array}\right] \tag{14}$$

$$[l_1(\mu)] = \begin{bmatrix} 1 & 0 & \mu \\ 0 & \mu & 1 \end{bmatrix}^{\mathrm{T}} \tag{15}$$

$$\{l_2(\mu)\} = \{1 \quad \mu^2 \quad 2\mu\}^{\mathrm{T}}. \tag{16}$$

Its determinant is an eighth degree polynomial in μ and in view of the positive definite stiffness matrix the roots μ_k must appear as four pairs of complex conjugates. If extension and bending are uncoupled, simplifications are obvious.

Williams-type stress singularities may now be obtained by taking

$$\phi_k(z_k) = \frac{z_k^{\delta+2}}{(\delta + 2)(\delta + 1)} \tag{17}$$

where δ is the order of the singularity (notation used by Wang and co-workers[4–6], whereas Ojikutu *et al.* denote the order by $\lambda = \delta + 1$). Since δ is in general a complex number, one should also consider its complex conjugate $\bar{\delta}$, but in view of the real parts taken in eqns (11) and (12), this is equivalent to the extension of the summation over the four remaining roots $\mu_{k+4} = \bar{\mu}_k$. The full expressions for displacements and stress resultants thus read

$$\{u\} = \mathrm{Re} \sum_{k=1}^{8} \{U_k\} \frac{z_k^{\delta+1}}{\delta+1} \tag{18}$$

$$w = \mathrm{Re} \sum_{k=1}^{8} W_k \frac{z_k^{\delta+2}}{(\delta+2)(\delta+1)} \tag{19}$$

$$\left\{\begin{array}{c} N \\ \hdashline M \end{array}\right\} = \mathrm{Re} \sum_{k=1}^{8} \left\{\begin{array}{c} N_k \\ \hdashline M_k \end{array}\right\} z_k^{\delta} \tag{20}$$

where

$$\left\{\begin{array}{c} N_k \\ M_k \end{array}\right\} = \left[\begin{array}{c:c} A & B \\ \hdashline B & D \end{array}\right] \left[\begin{array}{c:c} l_1(\mu_k) & 0 \\ \hdashline 0 & l_2(\mu_k) \end{array}\right] \left\{\begin{array}{c} U_k \\ W_k \end{array}\right\}. \tag{21}$$

The boundary conditions considered by Ojikutu *et al.* are

$$\begin{aligned} N_{x\theta} &= -N_{xx}\sin\theta + N_{xy}\cos\theta = 0 \\ N_{y\theta} &= -N_{xy}\sin\theta + N_{yy}\cos\theta = 0 \\ M_{\theta\theta} &= M_{xx}\sin^2\theta - 2M_{xy}\sin\theta\cos\theta + M_{yy}\cos^2\theta = 0 \\ w &= 0 \end{aligned} \tag{22}$$

on $\theta = \pm\alpha/2$, where α is the apex angle of the wedge. It may be noted that the range of the polar coordinate θ has been changed to $[-\alpha/2, +\alpha/2]$ with $\alpha < 2\pi$ in order to avoid multi-valued displacements arising from complex exponentiation in eqns (18) and (19). Substituting eqns (19) and (20) in eqn (22) after setting

$$z_k = r(\cos\theta + \mu_k \sin\theta) \tag{23}$$

and taking advantage of the fact that the eigenvectors of the stress resultants satisfy

$$\left[\begin{array}{c:c} l_1(\mu_k) & 0 \\ \hdashline 0 & l_2(\mu_k) \end{array}\right]^{\mathrm{T}} \left\{\begin{array}{c} N_k \\ \hdashline M_k \end{array}\right\} = \{0\} \tag{24}$$

one obtains a homogeneous system of eight complex algebraic equations

$$\sum_{k=1}^{8} (\cos\theta + \mu_k \sin\theta)^{\delta+1} \left\{\begin{array}{l} N_{xyk} \\ N_{yyk} \\ M_{yyk}\cos\theta + \dfrac{M_{xxk}}{\mu_k}\sin\theta \\ (\cos\theta + \mu_k\sin\theta) W_k \end{array}\right\} = 0 \tag{25}$$

where $\theta = -\alpha/2$, the unknowns being the arbitrary multiplicative complex constants of the eight eigenvectors (U_k, W_k, N_k, M_k). The order δ is then found as a root of the determinant of this system.

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